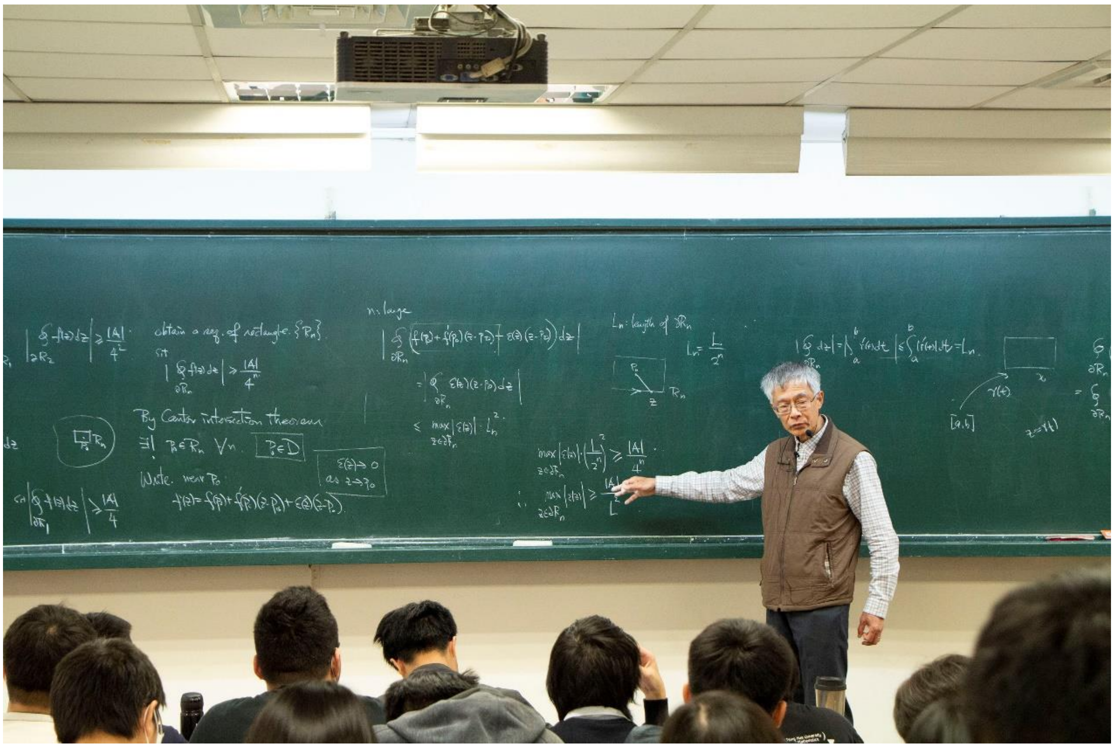
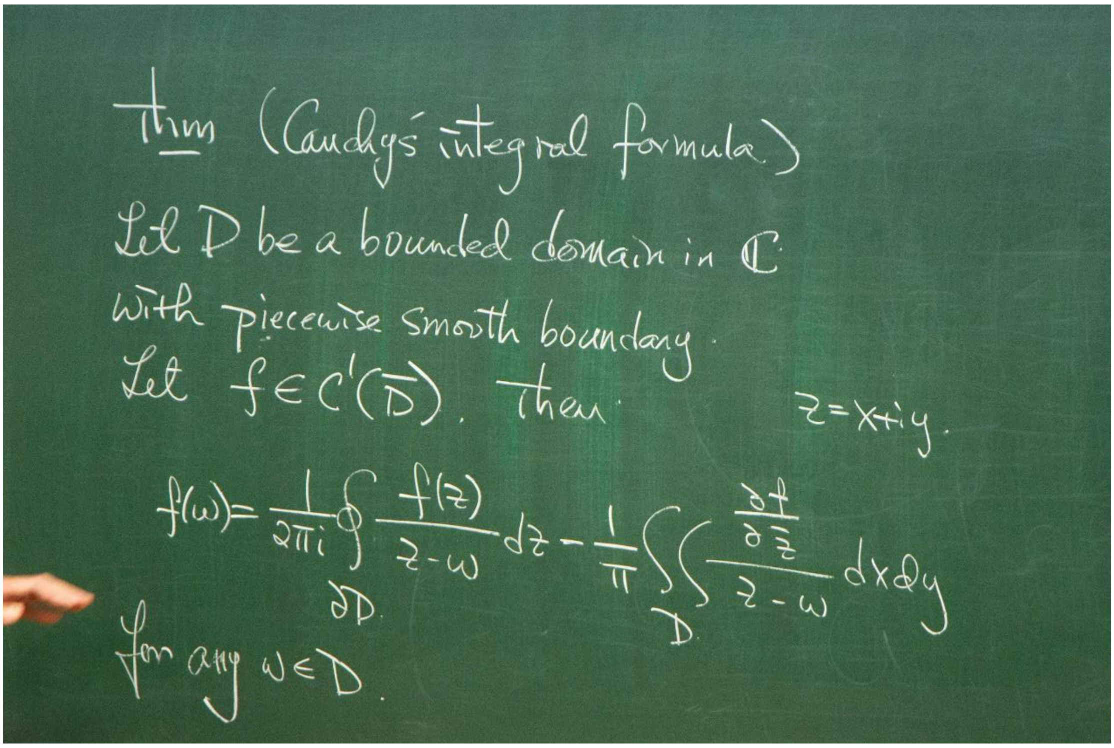
【10920 程守慶教授複變數函數論 / 第 2 堂版書】





thm (Cauchy's integral formula)

Let D be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary. Let $f \in C^1(\overline{D})$. Then

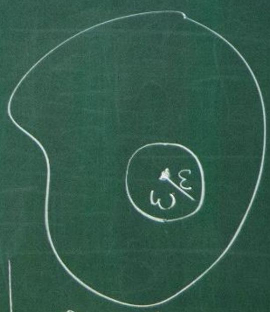
$z = x + iy$.

$$f(\omega) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z-\omega}\, dz - \frac{1}{\pi} \iint_D \frac{\frac{\partial f}{\partial \bar z}}{z-\omega}\, dx\,dy$$

for any $\omega \in D$.

Pf.

D



Consider

$D_\varepsilon = D \setminus \overline{B}_\varepsilon$

Pf.

D

Consider

$D_\varepsilon = D \setminus \overline{B}_\varepsilon$, $B_\varepsilon = B(\omega;\varepsilon)$

$$\oint_{\partial D_\varepsilon} \frac{f(z)}{z-\omega}\, dz \overset{\text{Green}}{=} 2i \iint_{D_\varepsilon} \frac{\partial}{\partial \bar z}\left(\frac{f(z)}{z-\omega}\right) dx\,dy = 2i \iint_{D_\varepsilon} \frac{\frac{\partial f}{\partial \bar z}(z)}{z-\omega}\, dx\,dy \qquad \frac{\partial f}{\partial \bar z} \in C(\overline{D})$$

$$\| $$

$$\oint_{\partial D} \frac{f(z)}{z-\omega}\, dz - \oint_{\partial B_\varepsilon} \frac{f(z)}{z-\omega}\, dz$$

$\|$ Set $z = \omega + \varepsilon e^{i\theta}$, $z - \omega = \varepsilon e^{i\theta}$

$$\int_0^{2\pi} \frac{f(\omega + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\, d\theta = i \int_0^{2\pi} f(\omega + \varepsilon e^{i\theta})\, d\theta \xrightarrow{\varepsilon \to 0} i \int_0^{2\pi} f(\omega)\, d\theta = 2\pi i f(\omega).$$

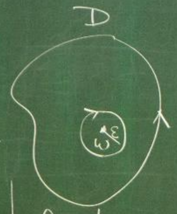
Pf.

Green

$$\oint_{\partial D_\varepsilon} \frac{f(z)}{z-w}\,dz = 2i\iint_{D_\varepsilon} \frac{\partial}{\partial \bar z}\left(\frac{f(z)}{z-w}\right)dxdy = 2i\iint_{D_\varepsilon} \frac{\frac{\partial f}{\partial \bar z}(z)}{z-w}\,dxdy \qquad \frac{\partial f}{\partial \bar z}\in C($$

$$\Big\| \qquad\qquad\qquad\qquad \downarrow \varepsilon\to 0$$

$$\oint_{\partial D}\frac{f(z)}{z-w}\,dz - \oint_{\partial B_\varepsilon}\frac{f(z)}{z-w}\,dz \qquad\qquad 2i\iint_D \frac{\frac{\partial f}{\partial \bar z}}{z-w}\,dxdy$$



xtiy.

|xdy|

Consider $D_\varepsilon = D\setminus \overline{B}_\varepsilon$, $B_\varepsilon = B(w;\varepsilon)$

Set $z = w+\varepsilon e^{i\theta}$, $z-w=\varepsilon e^{i\theta}$

$$\int_0^{2\pi} \frac{f(w+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta = i\int_0^{2\pi} f(w+\varepsilon e^{i\theta})\,d\theta \xrightarrow{\varepsilon\to 0} i\int_0^{2\pi} f(w)\,d\theta = 2\pi i f(w).$$

$$\frac{\frac{\partial f}{\partial \bar z}(z)}{z-w}\,dxdy \quad \downarrow \varepsilon\to 0 \quad \iint_D \frac{\frac{\partial f}{\partial \bar z}}{z-w}\,dxdy$$

$$\xrightarrow{\varepsilon\to 0} i\int_0^{2\pi} f(w)\,d\theta = 2\pi i f(w).$$

$\frac{\partial f}{\partial \bar z}\in C(\overline{D})$

Cauchy kernel

$$\iint_K \left|\frac{1}{z-w}\right| dxdy < \infty \qquad K:\text{compact}$$

$$\iint_B \frac{1}{|z|}\,dxdy = \int_0^{2\pi}\int_0^R \frac{1}{r}\, r\,dr\,d\theta$$

let $\varepsilon\to 0$

$$\oint_{\partial D}\frac{f(z)}{z-w}\,dz - 2\pi i f(w) = 2i\iint_D \frac{\frac{\partial f}{\partial \bar z}}{z-w}\,dxdy$$

$$f(w) = \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{z-w}\,dz - \frac{1}{\pi}\iint_D \frac{\frac{\partial f}{\partial \bar z}}{z-w}\,dxdy.$$

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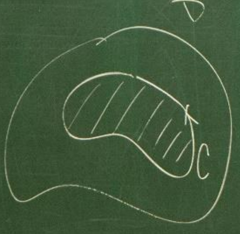
$= 2\pi i f(w).$

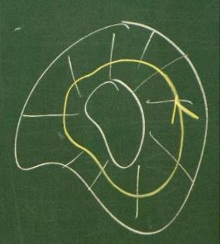
$$\iint_B \frac{1}{|z|}\,dx\,dy = \int_0^{2\pi}\int_0^R \frac{1}{r}\, r\,dr\,d\theta$$



Thm Let $D \subseteq \mathbb{C}$ be a simply-connected domain. Suppose $f \in C^1(D) \cap \mathcal{O}(D)$. Then

$$f(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-w}\,dz. \qquad w \in \Omega$$

Ω: region surrounded by the simple closed curve C.



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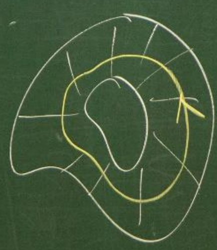

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Thm (Cauchy's)

Let D be a simply connected domain in $\mathbb{C}$.

C: Simple closed curve in D.

If $f \in \mathcal{O}(D)$. Then

$$\oint_C f(z)\, dz = 0$$

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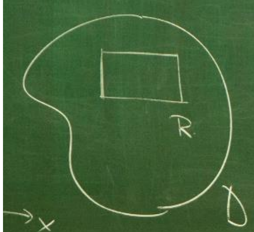
domain in $\mathbb{C}$.

pf. R: rectangle

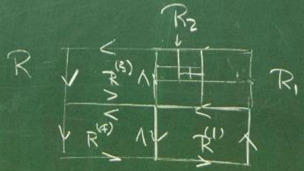
L: length of ∂R.

<u>claim</u>: $\oint_{\partial R} f(z)\,dz = 0$



Assume $\exists R$ s.t.

$$\oint_{\partial R} f(z)\,dz = A \neq 0$$



$$\oint_R f(z)\,dz = \sum_{k=1}^{4} \oint_{R^{(k)}} f(z)\,dz$$

$\therefore \exists R_1 = R^{(j)}$ some j s.t. $\left|\oint_{\partial R_1} f(z)\,dz\right| \geq \frac{|A|}{4}$

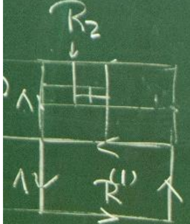
$\left|\oint_{\partial R_2} f(z)\,dz\right| \geq \frac{|A|}{4^2}$.

obta

s.t.

By Ca

$\exists!$ P_0



$$\left|\oint_{\partial R_2} f(z)\,dz\right| \geq \frac{|A|}{4^2}$$

obtain a seq. of rectangle. $\{R_n\}$.

s.t.

$$\left|\oint_{\partial R_n} f(z)\,dz\right| \geq \frac{|A|}{4^n}$$

By Cantor intersection theorem

$\exists!\ P_0 \in R_n\ \forall n$. $\boxed{P_0 \in D}$

$$\ldots)dz = \sum_{k=1}^{4} \oint_{R^{(k)}} f(z)\,dz$$

$R^{(j)}$ some j s.t. $\left|\oint_{\partial R_1} f(z)\,dz\right| \geq \frac{|A|}{4}$

R_1 $\left|\oint_{\partial R_2} f(z)\,dz\right| \geq \frac{|A|}{4^2}$

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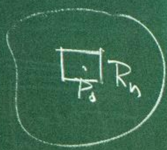
n: larg



Write. near P_0

$$f(z) = f(P_0) + f'(P_0)(z-P_0) + \varepsilon(z)(z-P_0)$$

$\boxed{\varepsilon(z) \to 0 \text{ as } z \to P_0}$

$\ldots f(z)\,dz$

s.t. $\left|\oint_{\partial R_1} f(z)\,dz\right| \geq \frac{|A|}{4}$

n: large

$$\oint_{\partial R_n} \left(\boxed{f(p_0) + f'(p_0)(z - z_0)} + \varepsilon(z)(z - z_0) \right) dz.$$

$\therefore\ f(p_0) + f'(p_0)(z - z_0) = Q'(z)$

$Q(z)$: polynomial of degree at most 2.

$\to 0$

$\to p_0$

n: large

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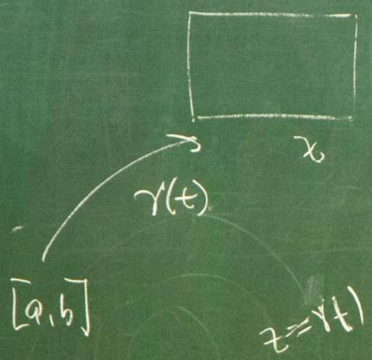
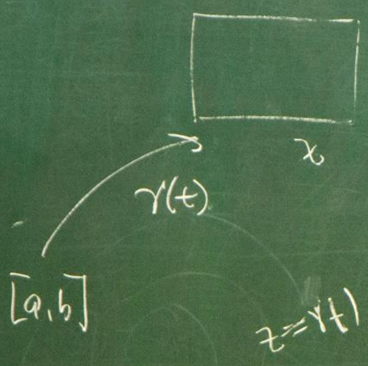


$$\oint_{\partial R_n} \left(f(p_0) + f'(p_0)(z-p_0) \right) dz$$

$$= \oint_{\partial R_n} Q'(z)\, dz = \int_a^b Q'(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_a^b dQ(\gamma(t))$$

$$= Q(\gamma(b)) - Q(\gamma(a)) = 0 .$$



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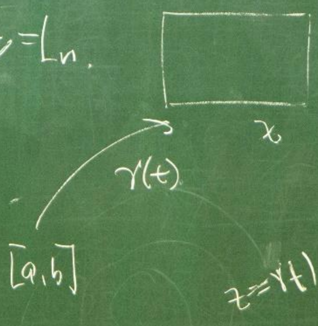
$$\left|\oint_{\partial R_n} dz\right| = \left|\int_a^b \gamma'(t)\,dt\right| \le \int_a^b |\gamma'(t)|\,dt = L_n.$$



$$\oint_{\partial R_n} \big(f(p_0)+f'(p_0)$$

$$= \oint_{\partial R_n} Q'(z)\,dz$$

$$=$$

$$= 0$$

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n: large

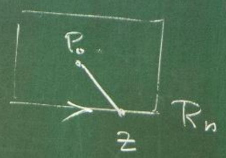
$$\left|\oint_{\partial R_n}\Big(f(p_0)+f'(p_0)(z-p_0)+\varepsilon(z)(z-p_0)\Big)\,dz\right|$$

$$= \left|\oint_{\partial R_n}\varepsilon(z)(z-p_0)\,dz\right|$$

$$\le \max_{z\in\partial R_n}|\varepsilon(z)|\cdot L_n^2.$$

L_n: length of ∂R_n

$$L_n = \frac{L}{2^n}$$



$\boxed{\begin{array}{l}\varepsilon(z)\to 0\\ z\to p_0\end{array}}$

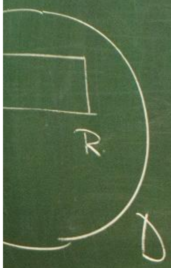
$$\max_{z\in\partial R_n}|\varepsilon(z)|\cdot\left(\frac{L}{2^n}\right)^2 \ge \frac{|A|}{4^n}.$$

$$\therefore\ \max_{z\in\partial R_n}|\varepsilon(z)| \ge \frac{|A|}{L^2} \Rightarrow 0 \ge \frac{|A|}{L^2}\ ※$$

pf. R: rectangle

L: length of ∂R.

$\mathbb{C}$. claim: $\oint_{\partial R} f(z)\,dz = 0$



let $z \in D$.

choose a reference point $z_0 \in D$.

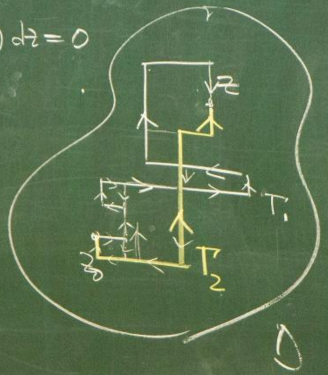
Well-defined

set

$$F(z) = \int_{\Gamma_1} f(z)\,dz$$

$$= \int_{\Gamma_2} f(z)\,dz$$

$\Gamma_1 - \Gamma_2$



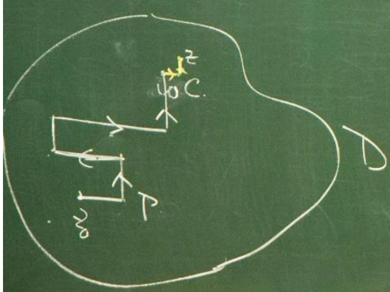
claim

$$F'(z) = f(z)$$

$$\lim_{z\to w} \frac{F(z)-F(w)}{z-w} = \lim_{z\to w} \frac{\int_{\Gamma+C} f(\eta)\,d\eta - \int_{\Gamma} f(\eta)\,d\eta}{z-w}$$

$$= \lim_{z\to w} \frac{\int_C f(\eta)\,d\eta}{z-w} = \lim_{z\to w} \frac{\int_C f(w)\,d\eta + \int_C (f(\eta)-f(w))\,d\eta}{z-w}$$

$$= f(w)$$



$$\frac{\int_{\Gamma+C} f(\eta)\,d\eta - \int_{\Gamma} f(\eta)\,d\eta}{z-w}$$

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$$= f(w)$$

$$\oint_C f(z)\,dz = \oint_C F'(z)\,dz$$

$$= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b d\,F(\gamma(t))$$

$$= F(\gamma(b)) - F(\gamma(a)) = 0$$

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Thm (Cauchy's integral formula)

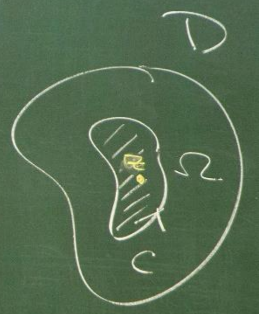
Let D be a simply-connected domain in $\mathbb{C}$.

C: simple closed curve in D.

$f \in \mathcal{O}(D)$. Let Ω be the region surrounded by C.

Then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} \, dw. \quad w \in \Omega.$$



捲積 Convolution

$$f * g(z) = \int f(w) \cdot g(z-w) \, dw$$

Thm (Cauchy's integral formula)

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domain in $\mathbb{C}$.

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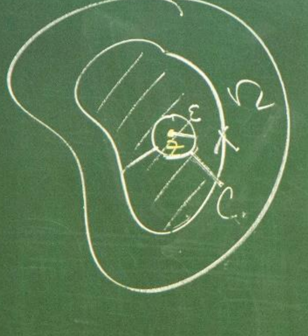
reproducing kernel

$\frac{1}{w-z}$: Cauchy kernel.

$\to x$



$$\Omega \setminus \overline{B}_\varepsilon = \Omega_\varepsilon.$$

$$\int_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}\,dw = 0.$$

$$\int_{\partial\Omega} \frac{f(w)}{w-z}\,dw - \int_{\partial B_\varepsilon} \frac{f(w)}{w-z}\,dw = 0.$$

$\partial\Omega = C.$

$$\int_{\partial B_\varepsilon} \frac{f(w)}{w-z}\,dw \xrightarrow{\varepsilon \to 0} 2\pi i f(z).$$